

Deformed Gauge Theories

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Abstract

Gauge theories are studied on a space of functions with the Moyal-Weyl product. The development of these ideas follows the differential geometry of the usual gauge theories, but several changes are forced upon us. The Leibniz rule has to be changed such that the theory is now based on a twisted Hopf algebra. Nevertheless, this twisted symmetry structure leads to conservation laws. The symmetry has to be extended from Lie algebra valued to enveloping algebra valued and new vector potentials have to be introduced. As usual, field equations are subjected to consistency conditions that restrict the possible models. Some examples are studied.

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1 Introduction

Gauge theories have been formulated and developed on the algebra of functions with a pointwise product:

$$\mu\{f \otimes g\} = f \cdot g. \quad (1.1)$$

This product is associative and commutative.

Recently, algebras of functions with a deformed product have been studied intensively [1]. These deformed (star-)products remain associative but not commutative.

The simplest example is the Moyal-Weyl product.

$$\mu_\star\{f \otimes g\} = \mu\{e^{\frac{i}{2}\theta^{\rho\sigma}\partial_\rho \otimes \partial_\sigma} f \otimes g\}. \quad (1.2)$$

It had its first appearance in quantum mechanics [2].

The star product can be seen as a higher order f -dependent differential operator acting on the function g . For the example of the Moyal-Weyl product this is

$$f \star g = \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{i}{2}\right)^n \theta^{\rho_1 \sigma_1} \dots \theta^{\rho_n \sigma_n} \left(\partial_{\rho_1} \dots \partial_{\rho_n} f\right) \partial_{\sigma_1} \dots \partial_{\sigma_n} \cdot g. \quad (1.3)$$

The differential operator maps the function g to the function $f \star g$.

The inverse map also exists [3], [4]. It \star -maps the function g to the function obtained by pointwise multiplying it with f

$$X_f^\star \star g = f \cdot g \quad (1.4)$$

For the Moyal-Weyl product we obtain

$$X_f^\star = \sum_{n=0}^{\infty} \frac{1}{n!} \left(-\frac{i}{2}\right)^n \theta^{\rho_1 \sigma_1} \dots \theta^{\rho_n \sigma_n} \left(\partial_{\rho_1} \dots \partial_{\rho_n} f\right) \star \partial_{\sigma_1}^\star \dots \partial_{\sigma_n}^\star. \quad (1.5)$$

The star-acting derivatives we denote by ∂_ρ^\star . For the Moyal-Weyl product the \star -derivatives and the usual derivatives are the same. In general this will not be the case. Star differentiation and star differential operators have been thoroughly discussed [4], [5].

In this lecture we are going to study gauge transformations on Moyal-Weyl or θ -deformed spaces.

2 Gauge transformations

Undeformed gauge transformations are Lie algebra valued:

$$\begin{aligned} \delta_\alpha \phi(x) &= i\alpha(x)\phi(x), \\ \alpha(x) &= \sum_l \alpha^l(x) T^l, \\ [T^l, T^k] &= i f^{lkr} T^r, \\ [\delta_\alpha, \delta_\beta] \phi &= [\alpha, \beta] \phi = -i \delta_{[\alpha, \beta]} \phi. \end{aligned} \quad (2.6)$$

The *deformed* gauge transformations [6], [7] are defined as follows:

$$\delta_\alpha^\star \phi = iX_\alpha^\star \star \phi = iX_{\alpha^l}^\star T^l \star \phi = i\alpha \cdot \psi. \quad (2.7)$$

From the fact that $X_f^\star \star X_g^\star = X_{f \cdot g}^\star$ we conclude:

$$[\delta_\alpha^\star ; \delta_\beta^\star] \phi = -i\delta_{[\alpha, \beta]}^\star \phi. \quad (2.8)$$

The \star -transformations represent the algebra via the \star -commutator.

Before we construct gauge theories we have to learn how products of fields transform.

In the *undeformed* situation we use, without even thinking, the Leibniz rule:

$$\delta_\alpha(\phi \cdot \psi) = (\delta_\alpha \phi) \cdot \psi + \phi \cdot (\delta_\alpha \psi) \quad (2.9)$$

and we can easily verify that this Leibniz rule is consistent with the Lie algebra:

$$[\delta_\alpha, \delta_\beta](\phi \cdot \psi) = -i\delta_{[\alpha, \beta]}(\phi \cdot \psi). \quad (2.10)$$

For the *deformed* transformation law of a \star -product of fields we demand a transformation law that is in the class of transformations defined in (2.7) [3], [4], [6], [8], [9]. This amounts to first decomposing the representation $\phi \star \psi$ for x -independent parameters into its irreducible parts and then follow (2.7) for gauging

$$\delta_\alpha^\star(\phi \star \psi) = iX_{\alpha^l}^\star \star \{T^l \phi \star \psi + \phi \star T^l \psi\}. \quad (2.11)$$

Certainly it is consistent with the Lie algebra:

$$[\delta_\alpha^\star ; \delta_\beta^\star](\phi \star \psi) = -i\delta_{[\alpha, \beta]}^\star(\phi \star \psi) \quad (2.12)$$

Because $\phi \star \psi$ is a function we can use the definition of X_f^\star given in (1.4) and simplify (2.11)

$$\delta_\alpha^\star(\phi \star \psi) = i\alpha^l \cdot \{T^l \phi \star \psi + \phi \star T^l \psi\}. \quad (2.13)$$

As α^l does not commute with the \star -operation this is different from (2.9). To see this difference more clearly we expand (2.13) in θ

$$\begin{aligned} \delta_\alpha^\star(\phi \star \psi) &= i\alpha^l \{T^l \phi \cdot \psi + \phi \cdot T^l \psi \\ &\quad + \frac{i}{2}\theta^{\rho\sigma} (T^l \partial_\rho \phi \cdot \partial_\sigma \psi + \partial_\rho \phi \cdot T^l \partial_\sigma \psi) + O(\theta^2)\}. \end{aligned} \quad (2.14)$$

The final version of the Leibniz rule for the \star -product should be entirely expressed with \star -operations. Thus we express (2.14) with \star -products. A short calculation shows:

$$\begin{aligned} \delta_\alpha^\star(\phi \star \psi) &= i(\alpha \phi) \star \psi + i\phi \star (\alpha \psi) \\ &\quad - \frac{i}{2}\theta^{\rho\sigma} \left(i((\partial_\rho \alpha^l) T^l \phi) \star (\partial_\sigma \psi) + (\partial_\rho \phi) \star i((\partial_\sigma \alpha^l) T^l \psi) \right) + O(\theta^2). \end{aligned} \quad (2.15)$$

With more work we can prove by induction to all orders in θ the following equation:

$$\begin{aligned}\delta_\alpha^*(\phi \star \psi) &= i \sum_{n=0}^{\infty} \frac{1}{n!} \left(-\frac{i}{2}\right)^n \theta^{\rho_1 \sigma_1} \dots \theta^{\rho_n \sigma_n} \{(\partial_{\rho_1} \dots \partial_{\rho_n} \alpha) \phi \star (\partial_{\sigma_1} \dots \partial_{\sigma_n} \psi) \\ &\quad + (\partial_{\rho_1} \dots \partial_{\rho_n} \phi) \star (\partial_{\sigma_1} \dots \partial_{\sigma_n} \alpha) \psi\}.\end{aligned}\quad (2.16)$$

This is different from what we obtain by putting just stars in the Leibniz rule (2.9). But this difference has a well-defined meaning if we use the Hopf algebra language to derive the Leibniz rule.

3 Hopf algebra techniques

The essential ingredient for a Hopf algebra [10] is the comultiplication $\Delta(\alpha)$: For the *undeformed* situation we define:

$$\Delta(\alpha) = \alpha \otimes 1 + 1 \otimes \alpha. \quad (3.1)$$

It allows us to write the Leibniz rule (2.9) in the Hopf algebra language:

$$\delta_\alpha(\phi \cdot \psi) = \mu\{\Delta(\alpha)\phi \otimes \psi\}. \quad (3.2)$$

In the *deformed* situation we use a twisted coproduct:

$$\begin{aligned}\Delta_{\mathcal{F}}(\alpha) &= \mathcal{F}(\alpha \otimes 1 + 1 \otimes \alpha)\mathcal{F}^{-1}, \\ \mathcal{F} &= e^{-\frac{i}{2}\theta^{\rho\sigma}\partial_\rho \otimes \partial_\sigma}.\end{aligned}\quad (3.3)$$

Here \mathcal{F} is a twist that has all the properties to define a Hopf algebra with $\Delta_{\mathcal{F}}(\alpha)$ as a comultiplication [11], [12], [13]. We can show that the transformation (2.16) can be written in the form

$$\delta_\alpha^*(\phi \star \psi) = i\mu_\star\{\Delta_{\mathcal{F}}(\alpha)\phi \otimes \psi\} \quad (3.4)$$

that defines the Leibniz rule in terms of the twisted comultiplication and the product μ_\star . To show this we start from equation (2.13) and write it with the explicit definition of the \star -product:

$$\begin{aligned}\delta_\alpha^*(\phi \star \psi) &= i\alpha^l \mu\{e^{\frac{i}{2}\theta^{\rho\sigma}\partial_\rho \otimes \partial_\sigma} (T^l \phi \otimes \psi + \phi \otimes T^l \psi)\} \\ &= i \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{i}{2}\right)^n \theta^{\rho_1 \sigma_1} \dots \theta^{\rho_n \sigma_n} \left(\alpha^l T^l (\partial_{\rho_1} \dots \partial_{\rho_n} \phi) (\partial_{\sigma_1} \dots \partial_{\sigma_n} \psi) \right. \\ &\quad \left. + (\partial_{\rho_1} \dots \partial_{\rho_n} \phi) \alpha^l T^l (\partial_{\sigma_1} \dots \partial_{\sigma_n} \psi)\right).\end{aligned}\quad (3.5)$$

This we now rewrite as follows

$$\begin{aligned}\delta_\alpha^*(\phi \star \psi) &= i\mu(\alpha \otimes 1 + 1 \otimes \alpha) e^{\frac{i}{2}\theta^{\rho\sigma}\partial_\rho \otimes \partial_\sigma} \phi \otimes \psi \\ &= i\mu\{e^{\frac{i}{2}\theta^{\rho\sigma}\partial_\rho \otimes \partial_\sigma} \cdot e^{-\frac{i}{2}\theta^{\rho\sigma}\partial_\rho \otimes \partial_\sigma} (\alpha \otimes 1 + 1 \otimes \alpha) e^{\frac{i}{2}\theta^{\rho\sigma}\partial_\rho \otimes \partial_\sigma} \phi \otimes \psi \\ &= i\mu_\star\{\Delta_{\mathcal{F}}(\alpha)\phi \otimes \psi\}.\end{aligned}\quad (3.6)$$

The Hopf algebra of gauge transformations can also be formulated with functional derivatives. We again start with the gauge transformation in the *undeformed* situation.

$$\begin{aligned}\delta_\alpha \phi_r(x) &= i\alpha_{rk}(x)\phi_k(x) \\ &= i \int dz \alpha_{jk}(z)\phi_k(z) \frac{\delta}{\delta\phi_j(z)} \cdot \phi_r(x).\end{aligned}\tag{3.7}$$

The fields ϕ can be in a reducible representation as well.

The generators of gauge transformations

$$S_\alpha = i \int dz \alpha_{jk}(z)\phi_k(z) \frac{\delta}{\delta\phi_j(z)}\tag{3.8}$$

can be considered as vector fields in the space of fields. They represent the algebra:

$$[S_\alpha, S_\beta] = -iS_{[\alpha, \beta]}.\tag{3.9}$$

A Hopf algebra structure can now be introduced via the coproduct:

$$\Delta(S_\alpha) = S_\alpha \otimes 1 + 1 \otimes S_\alpha.\tag{3.10}$$

It is easy to verify that it is consistent with the algebra:

$$[\Delta(S_\alpha), \Delta(S_\beta)] = -i\Delta(S_{[\alpha, \beta]})\tag{3.11}$$

and leads to the Leibniz rule

$$\mu\{\Delta(S_\alpha)\phi \cdot \psi\} = i\left((\alpha\phi) \cdot \psi + \phi \cdot (\alpha\psi)\right).\tag{3.12}$$

Again, we can deform the coproduct by a twist:

$$\mathcal{F} = e^{-\frac{i}{2}\theta^{\mu\nu} \int dz \partial_\mu \phi_l(z) \cdot \frac{\delta}{\delta\phi_l(z)} \otimes \int dy \partial_\nu \phi_k(y) \cdot \frac{\delta}{\delta\phi_k(y)}}\tag{3.13}$$

and define

$$\Delta_{\mathcal{F}}(S_\alpha) = \mathcal{F}^{-1} \Delta(S_\alpha) \mathcal{F}.\tag{3.14}$$

This twisted coproduct is again compatible with the Hopf algebra structure. When we derive the Leibniz rule from it

$$\delta_\alpha^*(\phi_r \star \phi_s) = \mu_\star\{\Delta_{\mathcal{F}}(S_\alpha)\phi_r \otimes \phi_s\}\tag{3.15}$$

we obtain (2.16). The Leibniz rules are identical.

The advantage of this formulation is that it is easy to include gauge fields as well. In the *undeformed* situation they are Lie algebra valued and transform as follows:

$$\delta A_\mu = \partial_\mu \alpha + i\alpha^l [T^l, A_\mu].\tag{3.16}$$

This gives rise to an additional term in the generator S_α :

$$S_\alpha^{A_\mu} = \int dz \{\partial_\mu \alpha^l(z) - \alpha^r(z) A_\mu^s(z) f^{rsl}\} \frac{\delta}{\delta A_\mu^l(z)}.\tag{3.17}$$

It generates the gauge transformations of $A_\mu^l(z)$ and it is consistent with the algebra relation (3.9). In the coproduct it has to be included and for the twist it demands an additional term as well:

$$\mathcal{F} = e^{-\frac{i}{2}\theta^{\mu\nu} \int dz \left(\partial_\mu \phi^l \frac{\delta}{\delta \phi_l} + \partial_\mu A_\rho^l \frac{\delta}{\delta A_\rho^l} \right) \otimes \int dy \left(\partial_\nu \phi^l \frac{\delta}{\delta \phi_l} + \partial_\nu A_\rho^l \frac{\delta}{\delta A_\rho^l} \right)}. \quad (3.18)$$

We can now calculate the contribution of the gauge field to the Leibniz rule. As an example we calculate:

$$\delta_\alpha^*(A_\mu \star \phi) = \mu^* \{ \Delta_{\mathcal{F}}(\alpha) A_\mu \otimes \phi \} \quad (3.19)$$

and obtain:

$$\begin{aligned} \delta_\alpha^*(A_\mu \star \psi) &= i\alpha^l \left([T^l, A_\mu] \star \psi \right) + i\alpha^l \left(A_\mu \star T^l \psi \right) + (\partial_\mu) \alpha^l T^l \psi \\ &= i\alpha^l T^l (A_\mu \star \psi) + (\partial_\mu) \alpha \psi. \end{aligned} \quad (3.20)$$

Now we can define a covariant derivative

$$D_\mu^* \psi = \partial_\mu \psi - i A_\mu \star \psi. \quad (3.21)$$

It will transform as usual if the vector field A_μ transforms as in (3.16):

$$\delta_\alpha^*(D_\mu^* \psi) = i\alpha^l T^l (D_\mu^* \psi) = iX_{\alpha^l}^* \star T^l (D_\mu^* \psi), \quad (3.22)$$

$$\delta_\alpha^* A_\mu = \partial_\mu \alpha + i\alpha^l [T^l, A_\mu] = \partial_\mu \alpha + iX_{\alpha^l}^* \star [T^l, A_\mu]. \quad (3.23)$$

From (3.23) we see that a Lie algebra valued vector field remains Lie algebra valued by the transformation (3.23).

4 Field equations

Now we proceed as in the *undeformed* situation. First we define the field strength tensor:

$$\begin{aligned} F_{\mu\nu} &= i[D_\mu^*, D_\nu^*] \\ &= \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu \star A_\nu]. \end{aligned} \quad (4.24)$$

Here we see already that $F_{\mu\nu}$ will not be Lie algebra valued even for Lie algebra valued vector fields.

Using the twisted gauge transformations we derive the transformation law of the field strength tensor:

$$\delta_\alpha^* F_{\mu\nu} = iX_{\alpha^l}^* \star [T^l, F_{\mu\nu}] = i[\alpha, F_{\mu\nu}]. \quad (4.25)$$

The expression $F^{\mu\nu} \star F_{\mu\nu}$ will transform accordingly.

The invariant Lagrangian we define as usual:

$$\mathcal{L} = \frac{1}{c} \text{Tr} (F^{\mu\nu} \star F_{\mu\nu}). \quad (4.26)$$

It is invariant and it is a deformation of the *undeformed* Lagrangian of a gauge theory.

To speak about an action we have to define integration. We take the usual integral over x and can verify that

$$\int d^4x f \star g = \int d^4x g \star f = \int d^4x f \cdot g \quad (4.27)$$

by partial integration. This is called the trace property of the integral.

Equation (4.27) allows a cyclic permutation of the fields under the integral. To derive the field equations we take the field to be varied to the very left. We work with the action

$$S = \frac{1}{c} \int d^4x \text{Tr} (F^{\mu\nu} \star F_{\mu\nu}). \quad (4.28)$$

From the trace property we compute:

$$\begin{aligned} \frac{\delta S}{\delta A_\rho(z)} &= \frac{2}{c} \int d^4x \text{Tr} \frac{\delta F_{\mu\nu}(x)}{\delta A_\rho(z)} \star F^{\mu\nu}(x) \\ &= \frac{2}{c} \int d^4x \text{Tr} \frac{\delta}{\delta A_\rho(z)} (\partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu \star A_\nu]) \star F^{\mu\nu}(x) \\ &= \frac{4}{c} \int d^4x \text{Tr} \frac{\delta}{\delta A_\rho(z)} (\partial_\mu A_\nu - iA_\mu \star A_\nu) \star F^{\mu\nu}(x) \end{aligned} \quad (4.29)$$

because $F^{\mu\nu}$ is antisymmetric. The last term is obtained after a cyclic permutation. The field equations are

$$\frac{\delta S}{\delta A_\rho(z)} = \frac{4}{c} \int d^4x \text{Tr} \frac{\delta A_\nu(x)}{\delta A_\rho(z)} \star (-\partial_\mu F^{\mu\nu}(x) + iA_\mu \star F^{\mu\nu} - iF^{\mu\nu} \star A_\mu). \quad (4.30)$$

These are exactly the equations we have expected from covariance:

$$D_\mu^\star F^{\mu\nu} = \partial_\mu F^{\mu\nu} - i[A_\mu \star F^{\mu\nu}] = 0. \quad (4.31)$$

We have already seen that $F_{\mu\nu}$ cannot be Lie algebra valued. From the field equations, considered as equations for the vector potential A_μ we see that A_μ cannot be Lie algebra valued either. We have to consider $F_{\mu\nu}$ and A_μ to be enveloping algebra valued. The additional vector field will introduce additional ghosts in the Lagrangian. To eliminate them we have to enlarge the symmetry to be enveloping algebra valued as well. For simplicity we assume α , A_μ and $F_{\mu\nu}$ to be matrix valued when the matrices act in the representation space of T^l .

From the field equations (4.31) follows a consistency equation because $F^{\mu\nu}$ is antisymmetric in μ and ν :

$$\partial_\nu [A_\mu \star F^{\mu\nu}] = 0. \quad (4.32)$$

To verify this condition we have to use the field equations:

$$\partial_\nu [A_\mu \star F^{\mu\nu}] = [\partial_\nu A_\mu \star F^{\mu\nu}] + [A_\mu \star \partial_\nu F^{\mu\nu}] \quad (4.33)$$

In the first term we replace $\partial_\nu A_\mu$ by $\frac{1}{2}(\partial_\nu A_\mu - \partial_\mu A_\nu)$ because $F_{\mu\nu}$ is antisymmetric in μ and ν . Then we express this term by $F_{\mu\nu}$ according to (4.24):

$$\frac{1}{2}(\partial_\nu A_\mu - \partial_\mu A_\nu) = \frac{i}{2}F_{\nu\mu} + \frac{i}{2}[A_\nu \star A_\mu]. \quad (4.34)$$

The $[F^{\mu\nu} \star F_{\mu\nu}]$ commutator vanishes and we are left with $\frac{i}{2}[[A_\nu \star A_\mu] \star F^{\mu\nu}]$ for the first term in (4.33). For the second term in (4.33) we use the field equations (4.31). Finally all terms left add up to zero if we use the Jacobi identity.

In all these equations A_μ and $F_{\mu\nu}$ are supposed to be matrices. We have suppressed the matrix indices.

A conserved current was found

$$j^\nu = [A_\mu \star F^{\mu\nu}], \quad \partial_\nu j^\nu = 0. \quad (4.35)$$

For $\theta = 0$ this is the current of a non-abelian gauge theory.

5 Matter fields

Matter fields can be coupled covariantly to the gauge fields via a covariant derivative. We start from a multiplet of the gauge group ψ_A not necessarily irreducible. The index A denotes the component of the field ψ in the representation space. The transformation law of ψ is: $\delta\psi_A = iX_{\alpha_{AB}}^* \psi_B = i\alpha_{AB}\psi_B$. For the usual gauge transformations α_{AB} will be Lie algebra valued. The covariant derivative is:

$$(D_\mu^* \star \psi)_A = \partial_\mu \psi_A - iA_{\mu AB} \star \psi_B. \quad (5.1)$$

The gauge potential A_μ is now supposed to be matrix valued in the representation space spanned by the matter fields.

For a spinor field

$$\bar{\psi}_{\alpha A} \gamma_{\alpha\beta}^\mu (D_\mu^* \star \psi)_A \quad (5.2)$$

will be invariant and therefore suitable for a covariant Lagrangian.

We consider the Lagrangian:

$$\mathcal{L} = \frac{1}{c} \text{Tr} F^{\mu\nu} F_{\mu\nu} + \bar{\psi} \star \gamma^\mu (i\partial_\mu + A_\mu \star) \psi - m \bar{\psi} \star \psi. \quad (5.3)$$

We have suppressed the matrix indices.

The field equations are obtained from (5.3) by varying the fields:

$$\frac{\delta \mathcal{L}}{\delta A_\rho} = \partial_\mu F_{AB}^{\mu\rho} + i[A_\mu \star F^{\rho\mu}]_{AB} + \gamma_{\alpha\beta}^\rho \psi_{\beta A} \star \bar{\psi}_{\alpha B} = 0 \quad (5.4)$$

and for the matter fields:

$$\begin{aligned} \frac{\delta \mathcal{L}}{\delta \bar{\psi}} &= \gamma^\mu (\partial_\mu \psi_A - iA_{\mu AB} \star \psi_B) + im \psi_A = 0 \\ \frac{\delta \mathcal{L}}{\delta \psi} &= (\partial_\mu \bar{\psi}_A \gamma^\mu + i\bar{\psi}_B \gamma^\mu \star iA_{\mu AB}) - im \bar{\psi}_A = 0. \end{aligned} \quad (5.5)$$

Again, equation (5.4) leads to a consistency relation that can be verified with the help of the field equations. It is, however, important that the representation space for the field ψ and the vector potential $A_{\mu AB}$ are the same. The representation space of the matter fields determines the space for the gauge potentials.

We conclude that there is a conserved current:

$$j_{AB}^\rho = i[A_\mu \star F^{\mu\rho}]_{AB} - \gamma_{\alpha\beta}^\rho \psi_{\beta A} \star \bar{\psi}_{\alpha B}. \quad (5.6)$$

We were again able to find a conserved current as a consequence of a deformed symmetry. Even if we put the vector potential to zero there remains the part from the matter field. There are conservation laws due to a deformed symmetry. It is remarkable that we have found conserved currents in the twisted theory as well. In the *undeformed* theory we can derive them with the help of the Noether theorem. In the deformed theory this is not possible. Nevertheless the property that a theory has a conserved current is preserved by a deformation. This is an important step to convince ourselves that a deformed gauge theory has properties close to what we need for physics.

6 Examples

1) Maxwell equations

We start from the simplest gauge theory based on U(1) and describing gauge fields only. We proceed schematically: The transformation law of the gauge field A_μ :

$$\delta_\alpha A_\mu = \partial_\alpha \quad (6.7)$$

The covariant derivative:

$$D_\mu^\star = (\partial_\mu - iA_\mu)^\star \quad (6.8)$$

The field strength tensor:

$$F_{\mu\nu} = [D_\mu^\star, D_\nu^\star] = \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu \star A_\nu] \quad (6.9)$$

The Lagrangian:

$$\mathcal{L} = -\frac{1}{4} F^{\mu\nu} \star F_{\mu\nu} \quad (6.10)$$

The field equations:

$$\partial^\mu F_{\mu\nu} - i[A^\mu \star F_{\mu\nu}] = 0 \quad (6.11)$$

Consistency equations:

$$\partial^\nu [A^\mu \star F_{\mu\nu}] = 0 \quad (6.12)$$

A schematic proof of the consistency condition:

$$[\partial^\nu A^\mu \star F_{\mu\nu}] + [A^\mu \star \partial^\nu F_{\mu\nu}] = \quad (6.13)$$

$$= \frac{i}{2} [[A^\nu \star A^\mu] \star F_{\mu\nu}] + i[A^\mu \star [A^\nu \star F_{\mu\nu}]] \quad (6.14)$$

We have used the field equations and the fact that $[F_{\mu\nu} \star F^{\mu\nu}] = 0$. The terms left can now be rearranged:

$$[[A^\nu \star A^\mu] \star F_{\mu\nu}] + [[A^\mu \star F_{\mu\nu}] \star A^\nu] + [[F_{\mu\nu} \star A^\nu] \star A^\mu] \quad (6.15)$$

and vanish due to the Jacobi identity.

We found a conserved current:

$$j_\nu = [A^\mu \star F_{\mu\nu}], \quad \partial_\nu j^\nu = 0. \quad (6.16)$$

2) Electrodynamics with one charged spinor field.

Transformation law of the gauge field and the spinor field:

$$\delta\psi = i\alpha\psi, \quad \delta A_\mu = \partial_\mu\alpha \quad (6.17)$$

Covariant derivative:

$$D_\mu^\star = (\partial_\mu - iA_\mu\star), \quad D_\mu^\star\psi = (\partial_\mu - iA_\mu\star)\psi \quad (6.18)$$

Field strength:

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu \star A_\nu] \quad (6.19)$$

Lagrangian:

$$\mathcal{L} = -\frac{1}{4}F^{\mu\nu} \star F_{\mu\nu} + \bar{\psi} \star \gamma^\mu (i\partial_\mu + A_\mu\star)\psi - m\bar{\psi} \star \psi \quad (6.20)$$

Field equations:

$$\begin{aligned} \partial_\mu F^{\mu\rho} + i[A_\mu \star F^{\rho\mu}] + \gamma^\rho\psi \star \bar{\psi} &= 0 \\ (\gamma^\mu(\partial_\mu - iA_\mu\star) + im)\psi &= 0 \\ \partial_\mu\bar{\psi}\gamma^\mu + i\bar{\psi}\gamma^\mu \star A_\mu - im\bar{\psi} &= 0 \end{aligned} \quad (6.21)$$

Consistency condition:

$$\partial_\rho([A_\mu \star F^{\rho\mu}] + \gamma^\rho\psi \star \bar{\psi}) = 0 \quad (6.22)$$

Proof: as before, the spinor terms have to be added in the current and the field equations.

Current:

$$j^\rho = [A_\nu \star F^{\rho\nu}] + \gamma^\rho\psi \star \bar{\psi}, \quad \partial_\nu j^\nu = 0. \quad (6.23)$$

3) Electrodynamics with several charged fields.

We try to formulate a model with one vector potential and differently charged matter fields as we do in the undeformed situation. This amounts to introduce an $U(1)$ gauge invariant action for the gauge potential and for the matter fields.

Let us consider the part of the vector potential first.

The transformation law is

$$\delta_\alpha A_\mu = \partial_\mu \alpha \quad (6.24)$$

The covariant derivative

$$D_\mu^\star = (\partial_\mu - iA_\mu \star) \quad (6.25)$$

gives the following field strength tensor

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu \star A_\nu]. \quad (6.26)$$

As an invariant Lagrangian we choose

$$\mathcal{L}_A = -\frac{1}{4} F^{\mu\nu} \star F_{\mu\nu}. \quad (6.27)$$

Next we consider the matter fields. They transform as follows

$$\delta_\alpha \psi^r = ig_r \alpha \psi^r. \quad (6.28)$$

The covariant derivative depends on the charge of the field it acts on:

$$D_\mu^\star \psi^r = (\partial_\mu - ig_r A_\mu \star) \psi^r. \quad (6.29)$$

The $U(1)$ gauge invariant action can be choosen as follows:

$$\mathcal{L}_\psi = \sum_r \bar{\psi}^r \star \gamma^\mu (i\partial_\mu + g_r A_\mu \star) \psi^r - m_r \bar{\psi}^r \star \psi^r. \quad (6.30)$$

As the total Lagrangian we take the sum

$$\mathcal{L} = \mathcal{L}_A + \mathcal{L}_\psi. \quad (6.31)$$

It is $U(1)$ gauge invariant and it is a deformation of the usual electrodynamics with different charged fields. This Lagrangian now leads to the field equations:

$$\begin{aligned} \partial_\mu F^{\mu\rho} + i[A_\mu \star F^{\rho\mu}] + \sum_r g_r \gamma^\rho \psi^r \star \bar{\psi}^r &= 0, \\ (\gamma^\mu (\partial_\mu - ig_r A_\mu \star) + im_r) \psi^r &= 0, \\ \partial_\mu \bar{\psi}^r \gamma^\mu + i\bar{\psi}^r \gamma^\mu \star g^r A_\mu - im_r \bar{\psi}^r &= 0. \end{aligned} \quad (6.32)$$

The first of these equations gives rise to a consistency condition:

$$\partial_\rho (i[A_\nu \star F^{\rho\nu}] + \sum_r g_r \gamma^\rho \psi^r \star \bar{\psi}^r) = 0. \quad (6.33)$$

From a direct calculation, using the field equations, follows:

$$\partial_\rho(i[A_\nu \star F^{\rho\nu}] + \sum_r g_r \gamma^\rho \psi^r \star \bar{\psi}^r) \quad (6.34)$$

$$= - \sum_r (g_r^2 - g_r) [A_\mu \star \gamma^\mu \psi^r \star \bar{\psi}^r]. \quad (6.35)$$

The consistency condition is only satisfied if $g_r = g_r^2$ or $g_r = 1$. With one vector potential we can in a U(1) model only describe particles with one charge. There can be an arbitrary number of matter fields with this charge. This is different from the usual undeformed situation. There the comutator in (6.26) vanishes and does not give rise to an inconsistency.

This is not surprising, we forgot that the vector potential has at least to be enveloping algebra valued. This is demonstrated in the next example.

4) Electrodynamics of a positive and a negative charged matter field.

The gauge group is supposed to be U(1) and the matter fields are in the multiplet that transforms as follows

$$\delta_\alpha \psi = i\alpha Q\psi, \quad Q = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (6.36)$$

As outlined in chapter 5, the gauge potential has to be in the same representation of the enveloping algebra as the matter fields are.

The enveloping algebra has two elements

$$I \text{ and } Q, \quad Q^2 = 1. \quad (6.37)$$

We generalize the transformation law (6.36) to be enveloping algebra valued

$$\delta_\Lambda = i\Lambda\psi, \quad \Lambda = \lambda_0(x)I + \lambda_1(x)Q. \quad (6.38)$$

The vector potential A_μ has the analogue decomposition

$$\mathcal{A}_\mu = A_\mu(x)I + B_\mu(x)Q. \quad (6.39)$$

The covariant derivative is

$$D_\mu^\star \psi = (\partial_\mu - i\mathcal{A}_\mu \star) \psi = (\partial_\mu - iA_\mu(x) \star I - iB_\mu(x) \star Q) \psi \quad (6.40)$$

The field strength can also be decomposed in the enveloping algebra

$$\mathcal{F}_{\mu\nu} = F_{\mu\nu}I + G_{\mu\nu}Q. \quad (6.41)$$

From the definition of the field strength

$$\mathcal{F}_{\mu\nu} = \partial_\mu \mathcal{A}_\nu - \partial_\nu \mathcal{A}_\mu - i[\mathcal{A}_\mu \star \mathcal{A}_\nu], \quad (6.42)$$

follows

$$\begin{aligned} F_{\mu\nu} &= \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu \star A_\nu] - i[B_\mu \star B_\nu], \\ G_{\mu\nu} &= \partial_\mu B_\nu - \partial_\nu B_\mu - i[A_\mu \star B_\nu] - i[B_\mu \star A_\nu]. \end{aligned} \quad (6.43)$$

The matter fields couple to the vector potential via the covariant derivative

$$\begin{aligned} D_\mu^\star \psi &= (\partial_\mu - i\mathcal{A}_\mu \star) \psi \\ &= (\partial_\mu - iA_\mu(x) \star I - iB_\mu(x) \star Q) \psi. \end{aligned} \quad (6.44)$$

This leads to the Lagrangian

$$\mathcal{L} = -\frac{1}{4} \mathcal{F}^{\mu\nu} \star \mathcal{F}_{\mu\nu} + \bar{\psi} \star \gamma^\mu (i\partial_\mu + \mathcal{A}_\mu \star) \psi - m \bar{\psi} \star \psi \quad (6.45)$$

and the field equations

$$\begin{aligned} \frac{\delta \mathcal{L}}{\delta A_\rho} : \quad & \partial_\mu F^{\mu\rho} + i[A_\mu \star F^{\rho\mu}] + i[B_\mu \star G^{\rho\mu}] + i\gamma^\rho \psi \star \bar{\psi} = 0, \\ \frac{\delta \mathcal{L}}{\delta B_\rho} : \quad & \partial_\mu G^{\mu\rho} + i[B_\mu \star F^{\rho\mu}] + i[A_\mu \star G^{\rho\mu}] + i\gamma^\rho \psi_A \star \bar{\psi}_B Q^{AB} = 0, \\ \frac{\delta \mathcal{L}}{\delta \bar{\psi}} : \quad & \gamma^\mu (\partial_\mu - i\mathcal{A}_\mu \star) \psi + m\psi = 0, \\ \frac{\delta \mathcal{L}}{\delta \psi} : \quad & \partial_\mu \bar{\psi} \gamma^\mu + i\bar{\psi} \gamma^\mu \star \mathcal{A}_\mu - m\bar{\psi} = 0. \end{aligned} \quad (6.46)$$

We obtain two consistency equations that render two transformation laws, in agreement with the extended symmetry (6.38)

$$j_A^\rho = i[A_\mu \star F^{\rho\mu}] + i[B_\mu \star G^{\rho\mu}] + \gamma^\rho \psi_A \star \bar{\psi}_A, \quad (6.47)$$

with

$$\partial_\rho j_A^\rho = 0 \quad (6.48)$$

and

$$j_B^\rho = i[B_\mu \star F^{\rho\mu}] + i[A_\mu \star G^{\rho\mu}] - i\gamma^\rho \psi_A \star \bar{\psi}_B Q^{AB}. \quad (6.49)$$

We learn that the deformed gauge theory leads to a theory with a larger symmetry structure, the enveloping algebra structure. This structure survives in the limit $\theta \rightarrow 0$. We find the corresponding conservation laws and gauge transformations needed for a consistent gauge theory.

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